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On the positive definiteness of the operator of the micropolar elasticity

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SUMMARY
The present paper is concerned with the static theory of anisotropic and inhomogeneous micropolar elastic solids. The operator of micropolar elasticity is considered and the positive definiteness of this operator for the first boundary value problem is proved. This fact leads to the existence of a generalized solution and to the applicability of the variational method [1] to this problem.

1. Introduction

In the last time some variational theorems and existence theorems in the linear theory of micropolar elasticity were derived. In the dynamic theory the general variational principles were established in [2–5]. In [6] the static theory of anisotropic and inhomogeneous micropolar elastic solids is considered. The existence and uniqueness of a weak solution of the boundary value problems is derived and some variational principles are established. Using the fundamental solutions, in [7], was proved the positive definiteness of the operator of micropolar elasticity for the first boundary value problem in the case of isotropic and homogeneous bodies. The existence theorems of the classical solutions are derived in [8, 9].

In this paper we consider the first boundary value problem in the static theory of anisotropic and inhomogeneous micropolar elastic solids. We prove, in a simple manner, that the operator of the micropolar elasticity is positive definite. From this fact it follows the existence of a generalized solution [1] and the applicability of the variational method developed in [1] to our problem.

2. Basic equations

Let us consider a finite region Ω of three-dimensional space, bounded by the surface S. For simplicity we assume the surface S to be piecewise smooth. Throughout this paper a rectangular coordinate system Ox_k $(k=1, 2, 3)$ is employed.

The basic equations in the static theory of anisotropic micropolar elastic solids are [10]

equilibrium equations

$$t_{ji,j}+f_i=0\,, \quad m_{ji,j}+\varepsilon_{ijk}t_{jk}+l_i=0\,, \tag{2.1}$$

constitutive equations

$$t_{ij}=A_{ijkl}e_{kl}+B_{ijkl}\varkappa_{kl}\,, \qquad m_{ij}=B_{klij}e_{kl}+C_{ijkl}\varkappa_{kl}\,, \tag{2.2}$$

geometrical equations

$$e_{ij}=u_{j,i}+\varepsilon_{jik}\varphi_k\,, \quad \varkappa_{ij}=\varphi_{j,i}\,. \tag{2.3}$$

In these equations we have used the following notations: t_{ij}: components of the stress tensor, m_{ij}: components of the couple-stress tensor, f_i: components of the body force vector, l_i: components of the body couple vector, u_i: components of the displacement vector, φ_i: components of the microrotation vector, e_{ij}, $\varkappa_{ij}$: kinematic characteristics of the strain, ε_{ijk}: alternating symbol, A_{ijkl}, B_{ijkl}, C_{ijkl}: characteristic coefficients of the material, the comma denotes partial derivation with respect to the variables x_i.

We have

$$A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij}. \tag{2.4}$$

We will assume that the coefficients $A_{ijkl}(x)$, $B_{ijkl}(x)$, $C_{ijkl}(x)$ are continuous and continuously differentiable in a closed region which contains the elastic medium.

The surface tractions and surface couples acting at point $x(x_k)$ on the surface S are given by

$$t_i = t_{ji} n_j, \quad m_i = m_{ji} n_j, \tag{2.5}$$

where n_j are the direction cosines of the outward normal to S at x.

3. The operator of the micropolar elasticity

From (2.1)–(2.3) we obtain the field equations of the static theory of micropolar elasticity in the form

$$A_i u = f_i, \quad A_{3+i} u = l_i, \tag{3.1}$$

where we used the notation

$$u = (u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3) \equiv (u_i, \varphi_i), \tag{3.2}$$

and we introduced the operators

$$\begin{aligned} A_i u &= -\frac{\partial}{\partial x_j}\left(A_{jikl}\frac{\partial u_l}{\partial x_k} + B_{jikl}\frac{\partial \varphi_l}{\partial x_k} + A_{jikl}\varepsilon_{lkm}\varphi_m\right), \\ A_{3+i} u &= -\frac{\partial}{\partial x_j}\left(B_{klji}\frac{\partial u_l}{\partial x_k} + C_{jikl}\frac{\partial \varphi_l}{\partial x_k} + B_{klji}\varepsilon_{lkm}\varphi_m\right) - \\ &\quad - \varepsilon_{ijk}A_{jkrs}\frac{\partial u_s}{\partial x_r} - \varepsilon_{ijk}B_{jkrs}\frac{\partial \varphi_s}{\partial x_r} - \varepsilon_{ijk}\varepsilon_{srm}A_{jkrs}\varphi_m. \end{aligned} \tag{3.3}$$

If we note

$$f = (f_1, f_2, f_3, l_1, l_2, l_3), \quad Au = (A_1 u, A_2 u, \ldots, A_6 u), \tag{3.4}$$

the system (3.1) can be written in the form

$$Au = f. \tag{3.5}$$

The differential operator A is the operator of the micropolar elasticity.

Let us consider the body subjected to two different systems of elastic loadings and the two corresponding states $C^{(\alpha)} = \{u_i^{(\alpha)}, \varphi_i^{(\alpha)}, e_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}, m_{ij}^{(\alpha)}\}$, $(\alpha = 1, 2)$. Using (2.1)–(2.5) and the divergence theorem we obtain

$$\int_S (t_i^{(1)} u_i^{(2)} + m_i^{(1)} \varphi_i^{(2)}) dx + \int_\Omega (f_i^{(1)} u_i^{(2)} + l_i^{(1)} \varphi_i^{(2)}) dx = 2\int_\Omega U_{12}\, dx, \tag{3.6}$$

where

$$\begin{aligned} 2U_{12} &= t_{ij}^{(1)} e_{ij}^{(2)} + m_{ij}^{(1)} \varkappa_{ij}^{(2)} = \\ &= A_{ijkl} e_{kl}^{(1)} e_{ij}^{(2)} + B_{ijkl}(e_{ij}^{(2)} \varkappa_{kl}^{(1)} + e_{ij}^{(1)} \varkappa_{kl}^{(2)}) + C_{ijkl} \varkappa_{kl}^{(1)} \varkappa_{ij}^{(2)}. \end{aligned} \tag{3.7}$$

If we consider the vectors

$$a = (a_1, a_2, \ldots, a_6), \quad b = (b_1, b_2, \ldots, b_6),$$

we will denote by ab the scalar product

$$ab = \sum_{i=1}^{6} a_i b_i. \tag{3.8}$$

If we denote

$$u=(u_i^{(1)},\varphi_i^{(1)}),\quad v=(u_i^{(2)},\varphi_i^{(2)}),\quad U_{12}=U(u,v),$$
$$p(u)=(t_i^{(1)},m_i^{(1)}),\quad p(v)=(t_i^{(2)},m_i^{(2)}),\tag{3.9}$$

the relation (3.6) can be written in the form

$$\int_S vp(u)dx+\int_\Omega vAudx=2\int_\Omega U(u,v)dx.\tag{3.10}$$

From (2.4), (3.7) it follows that

$$2U(u,v)=A_{ijkl}e_{kl}(u)e_{ij}(v)+B_{ijkl}(e_{ij}(v)\varkappa_{kl}(u)+e_{ij}(u)\varkappa_{kl}(v))+$$
$$+C_{ijkl}\varkappa_{kl}(u)\varkappa_{ij}(v)=2U(v,u),\tag{3.11}$$

so that from (3.10) we obtain

$$\int_\Omega(uAv-vAu)dx=\int_S[vp(u)-up(v)]dx.\tag{3.12}$$

If we denote $U(u)=U(u,u)$, from (3.10) we get

$$\int_\Omega uAudx=-\int_S up(u)dx+2\int_\Omega U(u)dx.\tag{3.13}$$

We assume that the internal energy density $U(u)$ is a positive definite quadratic form. Thus, we can write

$$2U(u)=A_{ijkl}e_{ij}e_{kl}+2B_{ijkl}e_{ij}\varkappa_{kl}+$$
$$+C_{ijkl}\varkappa_{ij}\varkappa_{kl}\geqslant c\sum_{i,j=1}^{3}(e_{ij}^2+\varkappa_{ij}^2),\quad(c>0,\quad c=\text{const.}).\tag{3.14}$$

The fact that in the linear theory the internal energy density is a positive definite quadratic form was intensively used in the classical theory of elasticity (see e.g. [1], [11–13]) and in the theory of elastic materials having a microstructure (see e.g. [14], [6], [7]).

We will consider the first boundary value problem of the micropolar elasticity theory, defined by the boundary condition

$$u\equiv(u_i,\varphi_i)=0\ \text{ on }\ S.\tag{3.15}$$

Let us consider the question of the uniqueness of the solution. The difference $u_0=(u_i^0,\varphi_i^0)$ of two solutions of the problem satisfies the homogeneous boundary condition and the homogeneous equation $Au_0=0$. From (3.13) we obtain

$$\int_\Omega U(u_0)dx=0.\tag{3.16}$$

By virtue of the positive definiteness of the form $U(u_0)$ it follows that

$$e_{ij}(u_0)=0,\quad \varkappa_{ij}(u_0)=0.\tag{3.17}$$

From (2.3), (3.17) we get

$$u_{i,j}^0+u_{j,i}^0=0,\quad \varphi_i^0=\tfrac{1}{2}\varepsilon_{imn}u_{n,m}^0,\quad \varphi_{i,j}^0=0,\tag{3.18}$$

so that

$$u_i^0=a_i+\varepsilon_{ijk}b_jx_k,\quad \varphi_i^0=b_i,\tag{3.19}$$

where a_i and b_i are arbitrary constants.

In the first homogeneous problem the boundary of the body is fixed so that

$$u_0=0\ \text{ in }\ \Omega.\tag{3.20}$$

4. The positive definiteness

Let us consider the equation

$$Au = f, \tag{4.1}$$

where A is a linear operator in a given real Hilbert space, and f and u are elements of the same space. It is known (see e.g. [1]) that if A is positive definite, then the equation (4.1) has a generalized solution which minimizes the functional

$$F(u) = (Au, u) - 2(u, f). \tag{4.2}$$

The domain of this functional is a new Hilbert space which is the closure of the domain D_A of the operator A in the metric generated by the scalar product (Au, u).

The operator A defined on the lineal D_A, which is dense in the considered Hilbert space, is called positive if

$$(Au, v) = (u, Av) \text{ for any } u, v \in D_A, \tag{4.3}$$

$$(Au, u) \geq 0, \qquad \text{for any } u \in D_A,$$

$$(Au, u) = 0 \text{ only holds for } u = 0. \tag{4.4}$$

The positive operator A is called positive definite if for any $u \in D_A$ the inequality

$$(Au, u) \geq \gamma^2 \|u\|^2, \tag{4.5}$$

holds, where γ is a positive constant.

In what follows we consider the operator of micropolar elasticity. We consider the real Hilbert space $L_2(\Omega)$, whose elements are vector-functions which are square summable in Ω; the scalar product in this space is defined by

$$(u, v) = \int_\Omega uv\,dx = \int_\Omega \sum_{i=1}^{3} (u_i v_i + \varphi_i \psi_i)\,dx, \tag{4.6}$$

where $u = (u_i, \varphi_i)$, $v = (v_i, \psi_i)$.

We denote by M_0 the set of vector functions $u(x) = (u_i, \varphi_i)$ in this space which are continuous and twice continuous differentiable in $\bar{\Omega}$ and vanish in some boundary layer Ω_δ; as usual, the width δ of the layer can depend upon u.

From (3.12) it follows that the operator of micropolar elasticity theory is symmetric on the set M_0. Moreover, using (3.13), (3.14) and (3.20) it follows that the operator A is positive on the set M_0.

Let us prove that the operator of micropolar elasticity theory is positive definite on the set M_0.

If $u \in M_0$, from (3.13) we have

$$(Au, u) = \int_\Omega uAu\,dx = 2\int_\Omega U(u)\,dx. \tag{4.7}$$

Using (3.14), (4.7), we get

$$(Au, u) \geq c \int_\Omega \sum_{i,j=1}^{3} (e_{ij}^2 + \varkappa_{ij}^2)\,dx. \tag{4.8}$$

Taking into account (2.3) we can write

$$e_{ij} = \gamma_{ij} + \varepsilon_{ijk}(\gamma_k - \varphi_k), \tag{4.9}$$

where

$$2\gamma_{ij} = u_{i,j} + u_{j,i}, \quad \gamma_k = \tfrac{1}{2}\varepsilon_{klm} u_{m,l}. \tag{4.10}$$

We have

$$\sum_{i,j=1}^{3} e_{ij}^2 = \sum_{i,j=1}^{3} \{\gamma_{ij}^2 + 2\gamma_{ij}\varepsilon_{ijk}(\gamma_k - \varphi_k) + [\varepsilon_{ijk}(\gamma_k - \varphi_k)]^2\} =$$

$$= \sum_{i,j=1}^{3} \{\gamma_{ij}^2 + [\varepsilon_{ijk}(\gamma_k - \varphi_k)]^2\} \geqslant \sum_{i,j=1}^{3} \gamma_{ij}^2 , \tag{4.11}$$

so that

$$(Au, u) \geqslant c \int_\Omega \sum_{i,j=1}^{3} [\gamma_{ij}^2 + (\varphi_{j,i})^2]\, dx . \tag{4.12}$$

Obviously, in our case the following lemma due to Friedrichs

$$\int_\Omega \sum_{i=1}^{3} u_i^2\, dx \leqslant c_1 \left\{ \int_\Omega \sum_{i,j=1}^{3} (u_{i,k})^2\, dx + \int_S u^2\, dx \right\}, \quad c_1 > 0 , \ c_1 = \text{const.} , \tag{4.13}$$

becomes

$$\int_\Omega \sum_{i=1}^{3} u_i^2\, dx \leqslant c_1 \int_\Omega \sum_{i,k=1}^{3} (u_{i,k})^2\, dx . \tag{4.14}$$

Using (4.14) and the first Korn's inequality [1]

$$\int_\Omega \sum_{i,k=1}^{3} (u_{i,k})^2\, dx \leqslant c_2 \int_\Omega \sum_{i,j=1}^{3} \gamma_{ij}^2 dx , \quad c_2 > 0 , \ c_2 = \text{const.} , \tag{4.15}$$

we obtain

$$\int_\Omega \sum_{i,j=1}^{3} \gamma_{ij}^2 dx \geqslant c_3 \int_\Omega \sum_{i=1}^{3} u_i^2\, dx ; \quad c_3 > 0 , \ c_3 = \text{const.} \tag{4.16}$$

If we apply the lemma (4.13), we get

$$\int_\Omega \sum_{i=1}^{3} \varphi_i^2\, dx \leqslant c_4 \int_\Omega \sum_{i,k=1}^{3} (\varphi_{i,k})^2\, dx ; \quad c_4 > 0 , \ c_4 = \text{const.} \tag{4.17}$$

From (4.12), (4.16) and (4.17) we find that

$$(Au, u) \geqslant c_0 \int_\Omega \sum_{i=1}^{3} (u_i^2 + \varphi_i^2)\, dx = c_0 \|u\|^2 ; \quad c_0 > 0 , \ c_0 = \text{const.} , \tag{4.18}$$

that is, the operator of micropolar elasticity theory is positive definite on the set M_0. This fact enables us to introduce in M_0 the scalar product

$$[u, v] = (Au, v) = \int_\Omega [A_{ijkl} e_{kl}(u) e_{ij}(v) + B_{ijkl}(e_{ij}(v) \varkappa_{kl}(u) +$$

$$+ e_{ij}(u) \varkappa_{kl}(v)) + C_{ijkl} \varkappa_{ij}(u) \varkappa_{kl}(v)]\, dx , \tag{4.19}$$

and then complete M_0 relative to this scalar product.

We denote this space by H_I and we consider the problem: to find a vector in H_I which satisfies the equations of micropolar elasticity theory. This problem leads [1] to the following variational problem: to find a vector in H_I which minimizes the functional

$$F(u) = [u, u] - 2(u, f) = \int_\Omega [A_{ijkl} e_{kl}(u) e_{ij}(u) +$$

$$+ 2B_{ijkl} e_{ij}(u) \varkappa_{kl}(u) + C_{ijkl} \varkappa_{ij}(u) \varkappa_{kl}(u) - 2u_i f_i - 2\varphi_i l_i]\, dx . \tag{4.20}$$

The solvability of this minimization problem follows from the positive definiteness of the operator A. The vector u which minimizes the functional (4.20) satisfies the equation (3.5) and the boundary conditions (3.15) in a certain generalized sense [1].

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